# SPRING 2025: MATH 590 EXAM 1 SOLUTIONS

**Name:**

Throughout $V$ will denote a vector space over $F = \mathbb{R}$ or $\mathbb{C}$. You must show all work to receive full credit.

**(I) True-False.** Write true or false next to each statement below. No explanation required. (3 points each)

(a) Any ten dimensional vector space can by spanned by thirteen vectors. True. One can simply add redundant vectors to a basis.

(b) The set of $2 \times 2$ matrices with trace equal to 5 form a subspace of $\mathrm{M}_{2\times 2}(\mathbb{R})$. False. The sum of two matrices with trace 5 would have trace 10.

(c) Any six vectors in $\mathbb{R}^6$ form a basis. False. Any six *linearly independent* vectors form a basis.

(d) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} a & b \\ e & f \end{pmatrix}$, then $|A+B| = |A| + |B|$. False. $\begin{vmatrix} a & b \\ c+e & d+f \end{vmatrix} = |A| + |B|$.

(e) Suppose $V = \mathrm{Span}\{v_1, v_2, v_3, v_4\}$ and $a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = \vec{0}$, with each $a_i \in F$ and $a_1 \neq 0$. Then $V = \mathrm{Span}\{v_2, v_3, v_4\}$. True. Just rewrite $v_1$ in terms of $v_2, v_3, v_4$.

**(II)** Carefully and **accurately** state the indicated definition, proposition or theorem. (10 points each)

(a) State the Exchange Theorem and be sure **define all terms used in your statement**.

Solution. **Exchange Theorem.** Let $w_1, \ldots, w_s, u_1, \ldots, u_r$ be vectors in $V$ and set $W := \mathrm{Span}\{w_1, \ldots, w_s\}$. Assume that $u_1, \ldots, u_r$ are linearly independent and belong to $W$. Then $r \leq s$. Moreover, after re-indexing the $w_i$'s, we have $W = \mathrm{Span}\{u_1, \ldots, u_r, w_{r+1}, \ldots, w_s\}$.

$\mathrm{Span}\{w_1, \ldots, w_s\}$ means the set of all linear combinations of $w_1, \ldots, w_s$ and $u_1, \ldots, u_r$ being linearly independent means that no $u_j$ belongs to the span of the set $\{u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_r\}$.

(b) State Cramer's Rule for an $n \times n$ system of linear equations.

Solution. **Cramer's Rule.** Suppose $A$ is an $n \times n$ matrix over $F$ and $A \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ is a system of linear equations such that $|A| \neq 0$. Let $B_i$ denote the matrix obtained from $A$ by replacing its $i$th column by $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$. Then, for each $1 \leq i \leq n$, $x_i = \frac{|B_i|}{|A|}$.

(c) For an $n \times n$ matrix over $\mathbb{R}$, state four conditions equivalent to $A$ being invertible.

Solution. The following statements are equivalent to the invertibility of $A$: $|A| \neq 0$; the rows of $A$ are linearly independent; the columns of $A$ are linearly independent; any system of linear equations with coefficient matrix $A$ has a unique solution.

(d) For an $n \times n$ matrix $A$, define the classical adjoint of $A$ and state its relevance to the inverse of $A$, if $A$ is invertible.

Solution. For each $1 \leq i \neq j \leq n$, let $A_{ij}$ denote the $(n-1) \times (n-1)$ matrix obtained from $A$ by deleting its $i$th row and $j$th column. Let $C$ denote the $n \times n$ matrix whose $i, j$th entry is $(-1)^{i+j}|A_{ij}|$. Then the classical adjoint of $A'$ of $A$ is $C^t$.

If $A$ is invertible, then $A^{-1} = \frac{1}{|A|} \cdot A'$.

**Calculation Problems.** (15 points each)

(a) Let $P_3(\mathbb{R})$ denote the vector space of polynomials having degree three or less. Set $p_1(x) = x^3 - 2x^2 + 2x - 4$ and $p_2(x) = 2x^3 + 6x^2 + 4x + 1$.

(i) Determine if $p_3(x) = 11x^3 + 18x^2 + 22x - 8$ belongs to $\text{Span}\{p_1(x), p_2(x)\}$. If so, write $p_3(x)$ as a linear combination of $p_1(x)$ and $p_2(x)$.

(ii) Extend the set $\{p_1(x), p_2(x)\}$ to a basis for $P_3(\mathbb{R})$.

Solution. We identify $p_1(x), p_2(x), p_3(x)$ with the column vectors $v_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \\ -4 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 6 \\ 4 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 11 \\ 18 \\ 22 \\ -8 \end{pmatrix}$. To determine if $p_3(x)$ is in $\text{Span}\{p_1(x), p_2(x)\}$, we determine if $v_3 \in \text{Span}\{v_1, v_2\}$. For this we use Gaussian elimination.

$$\left(\begin{array}{cc|c} 1 & 2 & 11 \\ -2 & 6 & 18 \\ 2 & 4 & 22 \\ -4 & 1 & -8 \end{array}\right) \xrightarrow{\text{EROs}} \left(\begin{array}{cc|c} 1 & 2 & 11 \\ 0 & 10 & 40 \\ 0 & 0 & 0 \\ 0 & 9 & 36 \end{array}\right) \xrightarrow{\text{EROs}} \left(\begin{array}{cc|c} 1 & 2 & 11 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{\text{ERO}} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

This shows $v_3 = 3v_1 + 4v_2$, and therefore, $p_3(x) = 3 \cdot p_1(x) + 4 \cdot p_2(x)$.

To extend $p_1(x), p_2(x)$ to a basis for $P_3(\mathbb{R})$, we first extend $v_1, v_2$ to a basis for $\mathbb{R}^4$. For this, we must find $v_3, v_4$ so that the matrix whose columns are $v_1, v_2, v_3, v_4$ are linearly independent. We try $v_3 = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $v_4 - e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Then,

$$\begin{vmatrix} 1 & 2 & 0 & 0 \\ -2 & 6 & 0 & 0 \\ 2 & 4 & 1 & 0 \\ -4 & 1 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 2 & 0 \\ -2 & 6 & 0 \\ 2 & 4 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 2 \\ -2 & 6 \end{vmatrix} = 10 \neq 0.$$

Thus, $v_1, v_2, v_3, v_4$ is a basis for $\mathbb{R}^4$, so that if we take $p_3(x) = x$ and $p_4(x) = 1$, $p_1(x), p_2(x), p_3(x), p_4(x)$ form a basis for $P_3(\mathbb{R})$.

Why did we try $e_3, e_4$? Note that the Gaussian elimination above shows that $\text{Span}\{v_1, v_2\} = \text{Span}\{e_1, e_2\}$ which suggests taking $v_3 = e_3$ and $v_4 = e_4$.

(b) Set $A = \begin{pmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{pmatrix}$.

(i) Find the characteristic polynomial for $A$ and the eignevalues of $A$.
(ii) For each eigenvalue, find a basis for the corresponding eigenspace.
(iii) Let $P$ be the matrix whose column vectors are the basis elements written in the order in which you found them. Find $P^{-1}$.
(iv) Verify that $P^{-1}AP$ is a diagonal matrix whose entries are the eigenvalues of $A$.

Solution. For (i), expanding along the last row we get,

$$p_A(x) = \begin{vmatrix} x-4 & 0 & 2 \\ -2 & x-5 & -4 \\ 0 & 0 & x-5 \end{vmatrix} = (x-5)\cdot\{(x-4)(x-5)\},$$

so the eigenvalues of $A$ are: 4, 5.

For (ii), $E_4$ is the null space of $\begin{pmatrix} 0 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. The vector $v_1 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ is a basis for this solution space, and hence a basis for $E_4$.

$E_5$ is the null space of the matrix $\begin{pmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The null space of this latter matrix has dimension two, and $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ are independent vectors in this null space and hence form a basis for $E_5$.

For (iii), we take $P = \begin{pmatrix} -1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. The usual gaussian elimination to find $P^{-1}$ yields $P^{-1} = \begin{pmatrix} -1 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}$.

For (iv)

$$P^{-1}AP = \begin{pmatrix} -1 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{pmatrix}\cdot\begin{pmatrix} -1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -4 & 0 & -8 \\ 10 & 5 & 20 \\ 0 & 0 & -5 \end{pmatrix}\cdot\begin{pmatrix} -1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

**Proof Problem.** Define elementary $2\times 2$ matrices and use elementary matrices to prove that $|AB| = |A|\cdot|B|$ for $2 \times 2$ matrices $A$ and $B$ such that $B$ is invertible. (15 points)

Solution. An elementary matrix is one obtained from the identity matrix by performing an elementary row operation on the identity matrix; or obtained from the identity matrix by applying a row operation. From our determinant rules, we have that if $E$ is an elementary matrix, then $|EA| = |E|\cdot|A|$. If $B$ is invertible, then there exist elementary matrices, $E_1, E_2, E_3, E_4$ (at most four, in the $2 \times 2$ case) such that $E_4E_3E_2E_1 = B$. Then

$$|BA| = |E_4E_3E_2E_1A| = |E_1|\cdot|E_2|\cdot|E_3|\cdot|E_4|\cdot|A| = |E_4E_3E_2E_1|\cdot|A| = |B|\cdot|A|.$$

Or Equivalently, using column operations such that $F_1, \ldots, F_4 = B$.

$$|AB| = |AF_1F_2F_3F_4| = |A|\cdot|F_1|\cdot|F_2|\cdot|F_3|\cdot|F_4| = |A|\cdot|F_1F_2F_3F_4| = |A|\cdot|B|.$$

**Optional Bonus Problems.** Solutions to bonus problems must be essentially completely correct to receive any credit. **Use the back of this page if necessary.**

1. Let $V$ be a finite dimensional vector space, and $W_1, W_2 \subseteq V$ subspaces. Prove that

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Hint: Start with a basis for $W_1 \cap W_2$. (10 points)

Solution. Suppose $u_1, \ldots, u_r$ is a basis for $W_1 \cap W_2$. Extend this to a basis $u_1, \ldots, u_r, w_1, \ldots, w_t$ for $W_1$ and a basis $u_1, \ldots, u_u, v_1, \ldots, v_s$ for $W_2$. Then $\dim(W_1) = r + t$ and $\dim(W_2) = r + s$. If we show that $B := \{u_1, \ldots, u_r, w_1, \ldots, w_t, v_1, \ldots, v_s\}$ is a basis for $W_1 + W_2$, then

$$\dim(W_1 + W_2) = r + s + t = (r + t) + (r + s) - r = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Suppose $a \in W_1 + W_2$. Then $a = b + c$, for some $a \in W_1$ and $b \in W_2$. We can write

$$b = \alpha_1 u_1 + \cdots + \alpha_r u_r + \beta_1 w_1 + \cdots + \beta_t w_1$$
$$c = \gamma_1 u_1 + \cdots + \gamma_r u_r + \delta_1 v_1 + \cdots + \delta_s v_s$$

Adding we see that $W_1 + W_2$ is spanned by $B$.

Now suppose

$$\alpha_1 u_1 + \cdots + \alpha_r u_r + \beta_1 w_1 + \cdots + \beta_t w_t + \delta_1 v_1 + \cdots + \delta_s v_s = \vec{0}. \qquad (*)$$

Then, $\alpha_1 u_1 + \cdots + \alpha_r u_r + \beta_1 w_1 + \cdots + \beta_t w_t = -\delta_1 v_1 - \cdots - \delta_s v_s$, so this vector belongs to $W_1 \cap W_2$. Therefore, we may write $-\delta_1 v_1 - \cdots - \delta_s v_s = \alpha'_1 u_1 + \cdots + \alpha'_r u_r$. Substituting into (*) we get

$$(\alpha_1 - \alpha'_1)u_1 + \cdots + (\alpha_r - \alpha'_r)u_r + \beta_1 w_1 + \cdots + \beta_t w_t = \vec{0}.$$

Since $u_1, \ldots, u_r, w_1, \ldots, w_t$ is a basis for $W_1$, each $\beta_i = 0$. Using this in (*), we have

$$\alpha_1 u_1 + \cdots + \alpha_r u_r + \delta_1 v_1 + \cdots + \delta_s v_s = \vec{0}.$$

Since these latter vectors are a basis for $W_2$, all $\alpha_i, \delta_j$ are 0, hence the set $B$ is linearly independent, and thus a basis for $W_1 + W_2$.

2. For $W_1, W_2, W_3 \subseteq V$, we write $V = W_1 \oplus W_2 \oplus W_3$, as a direct sum, if every element in $V$ can be written uniquely as a sum of elements from $W_1, W_2, W_3$. Show that, in this case: (i) $V = W_1 + W_2 + W_3$ and (ii) $W_i \cap (W_j + W_k) = \vec{0}$, for $1 \leq i \neq j \neq k \leq 3$. (10 points)

Solution. By assumption, every vector in $V$ is a sum of vectors from $W_1, W_2, W_2$. Suppose $v \in W_i \cap (W_j + W_k)$. Then $v = u + w$, with $u \in W_j$ and $w \in W_k$. Thus, $(-v) + u + w = \vec{0}$, with each $u, v, w$ coming from one of the given subspaces. On the other hand, $\vec{0} = \vec{0} + \vec{0} + \vec{0}$, with $\vec{0} \in W_1$, $\vec{0} \in W_2$, $\vec{0} \in W_3$. By uniqueness of sums, $v = \vec{0}, u = \vec{0}, w = \vec{0}$. In particular, $v = \vec{0}$, showing $W_i \cap (W_j + W_k) = \vec{0}$.